This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

QA
342
M14
SAL

ON THE DEFINITIONS

OF

THE TRIGONOMETRIC FUNCTIONS

BY

ALEXANDER MACFARLANE, M.A., D.Sc., LL.D.

FELLOW OF THE ROYAL SOCIETY OF EDINBURGH; PROFESSOR OF PHYSICS
IN THE UNIVERSITY OF TEXAS

ON THE DEFINITIONS

OF

THE TRIGONOMETRIC FUNCTIONS

BY

ALEXANDER MACFARLANE, M.A., D.Sc., LL.D.
Fellow of the Royal Society of Edinburgh; Professor of Physics
in the University of Texas

LIBRARY
LELAND STANFORD JUNIOR
UNIVERSITY

Norwood Press
J. S. CUSHING & CO., PRINTERS
BOSTON, U.S.A.

COPYRIGHT, 1894,
BY ALEXANDER MACFARLANE.

ON THE DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS.

[READ BEFORE THE MATHEMATICAL CONGRESS AT CHICAGO, AUGUST 22, 1893.]

IN a paper on "*The Principles of the Algebra of Physics*" I introduced a trigonometric notation for the partial products of two vectors, writing

$$\mathsf{AB} = \cos \mathsf{AB} + \mathrm{Sin}\, \mathsf{AB},$$

where cos AB denotes the positive scalar product, and Sin AB the directed vector product. To denote the magnitude of the vector product I used the notation sin AB without a capital: it is not the exact equivalent of the *tensor*, because the magnitude may be positive or negative. With the additional device of using the Greek letters α, β, γ, etc., to denote axes, it is possible to dispense with the peculiar symbols introduced into analysis by Hamilton, namely, S, V, T, U, K, I; and the space-analysis then assumes to a large extent the more familiar features of the ordinary analysis. The notation raises the question of the relation of space-analysis to trigonometry. If cos and sin are correct appellations of the products mentioned, are there products of two vectors which are correctly designated by tan, sec, cotan, cosec? At p. 87 of the *Principles* I give a brief answer to this question; but a complete answer called for a more thorough investigation than I had then time to make.

This trigonometrical notation has been briefly discussed by Mr. Heaviside (*The Electrician*, Dec. 9, 1892). He takes the position that vector algebra is far more simple and fundamental than trigonometry, and that it is a mistake to base vectorial notation upon that of a special application thereof of a more complicated nature. I believe that this paper will show that trigonometry is not an application of space-analysis, but an element of it; and that the ideas of this element are of the greatest importance in developing the higher elements of the analysis.

89220

The notation has also been discussed by Professor Alfred Lodge (*Nature*, Nov. 3, 1892). He takes the following view: "The particular symbol used to denote a scalar or a vector product is a matter of secondary importance, but is a matter which must sooner or later be settled if vector algebra is to come into general use. Lord Kelvin is of opinion that a function-symbol should be written with not less than three letters, and Professor Macfarlane's notation obeys that law, and is, moreover, easy to work with; but is incomplete, being applicable to products of two vectors only."

I consider that the notation is a matter not of secondary, but of paramount importance. If the notation is arbitrary, it gives us no help in the further development of analysis; if on the other hand it is systematic and logically connected with the existing notation of analysis, it points the way to more general principles and results. I believe that this paper will show that my notation is systematic and logical.

It is not true that the notation is applicable to products of only two vectors. In the *Principles* I have shown that the complete product of three vectors consists of three partial products, and that of four consists of five partial products: these several products are specified by means of the *cos* and *Sin* notation. The additional principle introduced is that in space of three dimensions the aspect of an area can be specified by the axis which it wants; hence that the complete product of an area-vector and a line-vector consists of two partial products which may be denominated the *cos* of the area and line, and the *Sin* of the area and line.

In this paper I propose first to review critically the historical definitions of the trigonometric terms, and the definitions, triangular, circular, or hyperbolic, given in the best modern treatises at my command; then to devise a logical system of definitions which will apply to space-analysis and that modern trigonometry which, as Professor Greenhill* shows, includes the properties both of circular and hyperbolic functions, and will be able to bring within the same domain the properties of the elliptic, general hyperbolic, and other functions. In this paper attention is mostly given to trigonometry in a plane; in a paper on *The Prin-*

---

* Differential and Integral Calculus, p. 61.

ciples of Elliptic and Hyperbolic Analysis I consider trigonometry in space.

The ancient method of defining the trigonometric terms is described by De Morgan at p. 18 of his "*Trigonometry and Double Algebra.*" A straight line OP of constant length (Fig. 1) revolves round O from a starting position OA; the arc AP traced out by the extremity of the revolving radius represents the angle AOP. From P draw a line PM perpendicular to OA; from A draw a line AT at right angles to OA, and terminating in OP produced; draw OB at right angles to OA and equal to OA, and from B draw BV at right angles to OB and terminating in the line of OP. The line PM is called the ***sine*** of the arc AP, the line OM is called the ***cosine***, the line AM the ***versed-sine***, the line AT the ***tangent***, the line OT the ***secant***, the line BV the ***cotangent***, and the line OV the ***cosecant***.

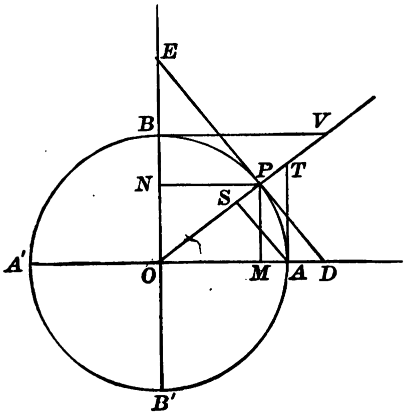

FIG. 1.

Here the terms sine, cosine, versed-sine, etc., are applied to certain lines drawn in and about a sector of a circle. These lines are commonly called the *trigonometric* lines; but inasmuch as they have reference to a circular sector and not to a triangle in general, they are more properly denominated *circular* lines. The trigonometric lines proper may be defined independently of the circle or any other curve.

We also remark that for the purposes of the higher analysis the circular lines must be defined with the utmost exactness;

difference of sense is not immaterial, still less is difference of direction. The sine-line is MP not PM, still less AS drawn from A perpendicular to OP. According to the account given by Dr. Hobson[*] of the ancient method, the tangent-line is not AT, but PD drawn a tangent to the circle at P, and terminated in the line of OA. Thus there are four logically distinct ways of defining the tangent line: *first,* it may mean the line drawn from A at right angles to OA; *second,* the line drawn from A a tangent to the circle at A; *third,* the line drawn from P at right angles to OP; *fourth,* the line drawn from P so as to touch the circle at P. The first definition agrees with the most ancient conceptions of the tangent; namely, the *umbra versa* of Abû'l Wafâ,[†] and the καθετος of Copernicus;[‡] the fourth view is taken by Professor Greenhill.[§] These four lines may be all unequal and differently directed when another curve such as the logarithmic spiral is substituted for the circle. It is necessary then to devise a separate notation for each.

In the same way there are four logically distinct definitions of the secant-line. It may mean, *first,* OT cut off by the perpendicular from A; *second,* OT cut off by the tangent at A; *third,* OD cut off by the perpendicular from P; *fourth,* OD cut off by the tangent at P. The first conception agrees with the ὑποτεινουσα of Copernicus,[||] while the fourth answers to the etymological conception of the tangent.

It is instructive to remember that the primary conception of the *sine* was the half of the chord of the double arc, and that it was long before the conception of the *cosine* was developed beyond that of the sine of the complementary arc.

The *circular ratios* are thus defined by De Morgan.[¶] Let θ denote the angle AOP (Fig. 1); then

$$\sin\theta = \frac{MP}{OP}, \quad \cos\theta = \frac{OM}{OP}, \quad \operatorname{vers}\theta = \frac{AM}{OP}, \quad \tan\theta = \frac{AT}{OA},$$

$$\sec\theta = \frac{OT}{OA}, \quad \operatorname{cotant}\theta = \frac{BV}{OB}, \quad \operatorname{cosec}\theta = \frac{OV}{OB}.$$

---

[*] Treatise on Plane Trigonometry, p. 16.

[†] Cantor's Vorlesungen über Geschichte der Mathematik, Vol. I., p. 642.

[‡] *Ibid.*, Vol. II., p. 433.

[§] Differential and Integral Calculus, p. 29.

[||] Cantor's Geschichte, Vol. II., p. 433.

[¶] Double Algebra, p. 19.

Here three different radii OA, OP, OB are introduced, but no reason is given why in a particular case one should be preferred to either of the others. Why should the secant be defined with respect to OA while the cosine is defined with respect to OP? Is it a matter of indifference which radius is taken? It may be as regards mere numerical ratios, but it is not so as regards geometric ratios. Accuracy of definition is essential to the higher development of trigonometry.

In consequence of defining some of the ratios with respect to the revolving line OP (Fig. 1) instead of the initial line OA, a difficulty in the signs is introduced; to wit, OP is always positive, even when coincident with OA' or OB', which are held to be negative. This view in my judgment partakes of the nature of a paradox. De Morgan attempts to dissolve it by the following explanation (*Double Algebra*, p. 8): —

"When the revolving line comes into the position OA', is it negative? I answer, no: OA' as a projection is considered as part of a line which makes an angle 0° with the starting-line; and on a line *so described* is negative. But OA' as a position of the line of revolution is part of a line which makes 180° with the starting-line; and thus considered it is positive. The same considerations apply to the other axis. A line may be considered as making *with itself* an angle of 0° or an angle of 180°; whatever signs its parts have in the first case they have the opposite ones in the second."

Now the terms *positive* and *negative*, symbolized by $+$ and $-$ respectively, are essentially relative; they in their simplest application compare one line with another. If the line compared has the same direction as the line of reference, it is positive with respect to that line; if it has the opposite direction, it is negative with respect to that line. The line OA' is negative with respect to OA, and it is equally true that OP when coincident with OA' is negative with respect to OA. The line OB' is negative with respect to OB, and OP when coincident with OB' is negative with respect to OB. There is no meaning in saying that OP is always positive. The fact is that we cannot dispense with the idea of an initial line as a basis of reference, and I propose to show in the development which follows that the ratios are properly defined with respect to this initial line. The radius which should appear in each of the definitions is the radius OA.

The modern method seeks to define the trigonometric ratios independently of the circle merely by means of two intersecting lines. In elementary works this is first done under the limitation that the lines intersect at an acute angle. For instance, Todhunter proceeds thus (*Plane Trigonometry*, p. 14): —

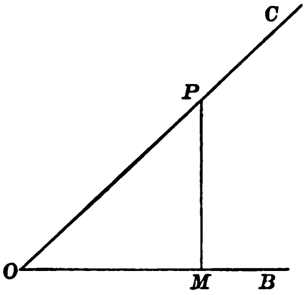

FIG. 2.

"Let BOC (Fig. 2) be any angle; take any point in either of the containing straight lines, and from it draw a perpendicular to the other straight line; let P be the point in the straight line OC, and PM perpendicular to OB. We shall use the letter A to dênote the angle BOC. Then

$\frac{PM}{OP}$, that is $\frac{\text{perpendicular}}{\text{hypotenuse}}$, is called the *sine* of the angle A;

$\frac{OM}{OP}$, that is $\frac{\text{base}}{\text{hypotenuse}}$, is called the *cosine* of the angle A;

$\frac{PM}{OM}$, that is $\frac{\text{perpendicular}}{\text{base}}$, is called the *tangent* of the angle A;

$\frac{OM}{PM}$, that is $\frac{\text{base}}{\text{perpendicular}}$, is called the *cotangent* of the angle A;

$\frac{OP}{OM}$, that is $\frac{\text{hypotenuse}}{\text{base}}$, is called the *secant* of the angle A;

$\frac{OP}{PM}$, that is $\frac{\text{hypotenuse}}{\text{perpendicular}}$, is called the *cosecant* of the angle A.

If the cosine of A be subtracted from unity, the remainder is called the *versed-sine* of A. If the sine of A be subtracted from unity, the remainder is called the coversed-sine of A." Equivalent definitions are given by Levett and Davison* and by Hobson.†

The definitions quoted are accurate only so far as arithmetical magnitude is concerned; they take no account of sense or direction. For exact purposes it is not indifferent whether the perpendicular be drawn from OB or from OC, and whether the sine be defined as $\frac{PM}{OP}$ or $\frac{MP}{OP}$. In consequence of dropping out the idea of an initial line it is necessary to compare OM and MP with OP, which does not coincide with the axis on which the projection is made. The cotangent so defined answers to the old conception of the *umbra*, the tangent to that of the *umbra versa*, and the secant to that of the *hypotenuse* of Copernicus. A difficulty is encountered with the *versed-sine;* for it is not defined geometrically like the others, as the ratio of two lines; it is defined analytically. Why this breakdown in the scheme of definitions? But the above definitions are not comprehensive enough even for the simple case where the lines meet at an obtuse angle, because then the triangle POM encloses not the angle BOC, but its supplement.

---

* Plane Trigonometry, p. 4.

† Plane Trigonometry, p. 16.

The definitions are extended by dropping the idea of a right-angled triangle, and substituting the idea of projection. Thus Levett and Davison, following De Morgan, say (p. 93): —

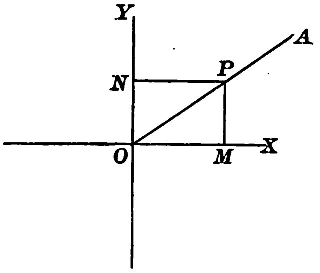

FIG. 3.

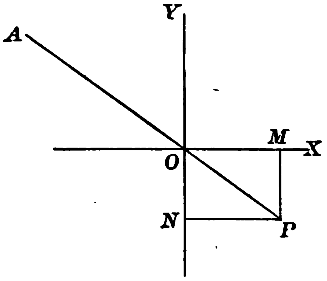

FIG. 4.

"Let a line rotate about O (Figs. 3 and 4) from OX through any positive or negative angle α to the position OA; let OY be a line making an angle $\frac{\pi}{2}$ in the positive sense with OX; and let OA, OX, OY be the positive senses of the lines OA, OX, OY. Let a length OP, of any magnitude and of either sense, be measured along OA; and let OM, ON be the projections of OP on OX, OY respectively. The ratio $OM : OP$ is called the *cosine* of the angle α, $ON : OP$ the *sine* of α, $ON : OM$ the *tangent* of α, $OP : OM$ the *secant* of α, $OP : ON$ the *cosecant* of α, and $OM : ON$ the *cotangent* of α. These ratios are called the Circular Functions of the angle α." The following is added in small print: "Two other ratios are occasionally used, and are defined as follows: If the length OP be equal in magnitude to OX, and positive in sense, and if $OY = OX$, the ratio $MX : OP$ is called the ***versine*** of α, and $NY : OP$ the ***coversine*** of α."

The above mode of defining assumes that a line may be positive in itself, whereas there are reasons for believing that positive and negative have their primary meaning in the comparison of two lines. Again, in order to define the versine, the two intersecting lines are given up, and conditions are imposed equivalent to introducing the circle; for OP is made of constant length, and is supposed to be always positive.

Mr. Carr in his *Synopsis of Pure Mathematics* defines the sine, cosine, and tangent geometrically; but the secant, cosecant, and cotangent as the respective reciprocals of these. It is surely more logical to define each function geometrically and independently, and afterwards prove what relations exist between them.

From the definitions examined we may conclude that under the one name of trigonometric ratios are comprised two species: the geometric, or rather *triangular*, and the *circular* proper. The triangular ratios are defined independently of the circle, and they include some of the circular ratios as special cases.

Further light on this subject may be obtained by considering those functions analogous to the circular which depend on the equilateral hyperbola, or ex-circle. The convenient terms "ex-circle" and "ex-circular" have been introduced by Mr. Hayward for the phrases "equilateral hyperbola" and "equilateral hyperbolic," commonly called "hyperbolic" (*Vector Algebra and Trigonometry*, p. 128). The following method of defining these ratios is adopted by Messrs. Levett and Davison (*Plane Trigonometry*, p. 258):—

"Let a point move along the curve (Fig. 5) from the vertex A of one branch of a rectangular hyperbola, whose centre is O and semi-axis equal to a, to the position P; let A be the area of the hyperbolic sector AOP, and let $u = \frac{2A}{a^2}$; that is, let u be the measure of the sector AOP, the unit of measurement being the square whose diagonal is the semi-axis.

"Take OY, a line making an angle of 90° in the positive sense with the transverse axis OAX, and let OM, ON be the projections of OP on OX, OY respectively; then the ratio

$OM : OA$ is called the *hyperbolic cosine* of u,
$ON : OA$ the *hyperbolic sine* of u,
$ON : OM$ the *hyperbolic tangent* of u,
$OA : OM$ the *hyperbolic secant* of u,
$OA : ON$ the *hyperbolic cosecant* of u,
$OM : ON$ the *hyperbolic cotangent* of u."

We observe that here the ratios are not defined with respect to the radius-vector OP, but with respect to OA the initial line; to define them with respect to OP would be an error. Wherefore, we conclude that it is the analogue of OA, not the analogue of OP, which should be introduced into the definitions of the circular ratios. We also observe that the hyperbolic argument is not the ratio of the arc to the initial radius, but the ratio of twice the area of the sector to the square on the initial radius; hence the true analytical argument for the circular ratios is not the ratio of the arc to the radius, but the ratios of twice the area of the sector to the square on the radius. This leads us to the idea that the trigonometric ratios may be ratios of areas as well as ratios of lines.

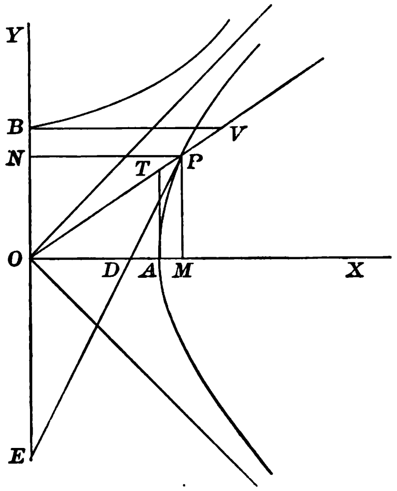

FIG 5.

Dr. Günther,[*] following M. Laisant,[†] gives the definitions of the circular lines which appear to furnish most readily the definitions of the analogous ex-circular lines. Let APB (Fig. 1) be a circle of unit radius, and let u denote double the area of the sector AOP; draw PM perpendicular to OA, and PN to OB; draw AT a tangent to the circle from A terminating in OP produced, and BV a tangent to the circle at B also terminating in OP produced; draw a tangent to the circle at P cutting the axis of OA in D, and that of OB in E. Then the line PM or ON represents $\sin u$, the line OM or NP $\cos u$; AT represents $\tan u$, and BV $\operatorname{cotan} u$; while OD, not OT, represents $\sec u$, and OE, not OV, $\operatorname{cosec} u$. The six ratios are represented by lines along the axes of projection, — three along the axis of abscissæ, and three along the axis of ordinates; none have the direction of the radius-vector. The definition of the tangent takes the second view, while that of the secant takes the fourth view of it mentioned at page 4 above.

The analogous lines are defined in the following manner: Let APB (Fig. 5) be an equilateral hyperbola of unit semi-diameter, and let u denote double the area of the sector AOP; draw PM perpendicular to OA, and PN to OB; draw AT a tangent to the hyperbola at A terminating in OP, and BV a tangent to the conjugate hyperbola at B also terminating in OP; draw a tangent to the hyperbola at P cutting the axis of OA in D, and that of OB in E. Then the line MP represents $\sinh u$, OM $\cosh u$, AT $\tanh u$, BV $\coth u$, OD $\operatorname{sech} u$, and OE $\operatorname{cosech} u$. The analogous ratios are represented by the analogous lines. We observe that AT and BV might have been defined as drawn at right angles to OA and OB respectively, that is, according to the first view of the tangent; but that OD corresponds to the fourth view of the secant, and to it only. Why is it that analysts find it easier to deal with lines which have the directions of the axes than with lines having any other direction such as that of the radius-vector, or of the true tangent? Because the former involve scalar products only, while the latter involve vector products.

M. Laisant, in his admirable *Essai*, extends his definitions of the trigonometric lines to the ellipse and general hyperbola.[‡]

---

[*] Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunctionen, p. 92.

[†] Essai sur les fonctions hyperboliques.

[‡] Essai sur les fonctions hyperboliques, p. 269.

Let APB' (Fig. 13) be an ellipse of such size that the product of its two semi-axes OA and OB' is unity. By u is meant twice the area of the sector AOP; elliptic $\cos u$ is represented by OM, elliptic $\sin u$ by MP, elliptic $\sec u$ by OD, elliptic $\tan u$ by AT, elliptic $\operatorname{cotan} u$ by $B'V$, and elliptic $\operatorname{cosec} u$ by OE. Here the denominator of the ratio u is the product of the two semi-axes.

Many analysts hold that the circular functions might be defined by purely algebraic ideas. For instance, De Morgan (*Double Algebra*, p. 34): "I said that we should soon make it very evident that a purely algebraical basis *might* have been made for trigonometry. If we had chosen to call the preceding functions of z, namely,

$$1-\frac{z^2}{2!}+, \qquad z-\frac{z^3}{3!}+, \qquad z+\frac{z^3}{3}+,$$

by the names of *cosine, sine,* and *tangent* of z (and their reciprocals *secant, cosecant,* and *cotangent*), we might have investigated the properties of these series, and we should at last have arrived at all our preceding formulæ of connection; but with much more difficulty."

Again, Dr. Hobson (*Plane Trigonometry*, p. 279): "It is possible to give purely analytical definitions of the circular functions, and to deduce from these definitions their fundamental analytical properties, so that the calculus of circular functions can be placed upon a basis independent of all geometrical considerations; these definitions will include the circular functions of a complex quantity. We can define the cosine and sine of z by means of the equations

$$\cos z = \tfrac{1}{2}\{e^{iz}+e^{-iz}\},$$

$$\sin z = \frac{1}{2i}\{e^{iz}-e^{-iz}\},$$

where e^z denotes the series $1+z+\frac{z^2}{2!}+$, etc. In other words, we define $\cos z$ as the sum of the series $1-\frac{z^2}{2!}+\frac{z^4}{4!}-$, and $\sin z$ as the sum of the series $z-\frac{z^3}{3!}+\frac{z^5}{5!}-$. We may regard this then as the generalized definition of the cosine and sine functions, and it includes the case of a complex argument, which was not included in the earlier geometric definitions."

A definition which has only an algebraic basis is, in my opinion, of the species which logicians call *nominal;* while one which has a geometrical basis is of the species called *real.* It may be doubted whether nominal definitions are of much scientific value. The primary geometric idea which is the basis of the primary trigonometric function can also be generalized, and in more ways than one; how can the analyst secure a correspondence between his arbitrarily generalized definition and the more general ideas which develop from the primary geometrical idea? In the present paper and in a paper on "*The Principles of Elliptic and Hyperbolic Analysis*" I show that there are several geometrically real generalizations of the circular functions, and that the algebraic series for the simple functions generalize in ways that would never be deduced by taking the elementary series as the general definitions.

I now proceed to consider how the several species of trigonometric functions — the triangular, the circular, and the ex-circular, — may be defined in harmony with one another. The method adopted is afterwards shown to be applicable to the logarithmic spiral, ellipse and general hyperbola, and to a mixed curve composed partly of a circle, partly of an ex-circle; further, in the paper on "*The Principles of Elliptic and Hyperbolic Analysis,*" it is applied to ellipsoidal and hyperboloidal trigonometry.

THE TRIANGULAR FUNCTIONS.

Let OA and OP represent (Fig. 6) any two finite straight lines, or vectors, meeting at the point O. A triangle is formed

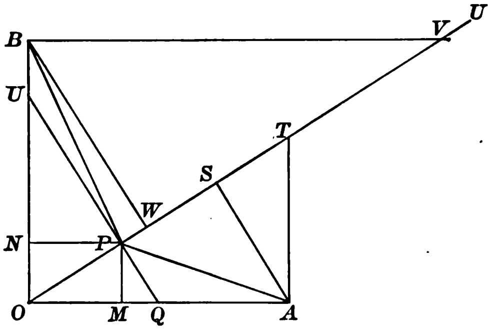

FIG. 6.

by joining A and P. From P draw PM at right angles to OA, and PQ at right angles to OP; from A draw AT at right angles to OA, and AS at right angles to OP.

First: we consider OM and MP the orthogonal projections of OP on OA. In a certain sense

$$OP = OM + MP;$$

to wit, in the ordinary sense of a vector equation. By prefixing OA to each term, we derive an area equation

$$(OA)(OP) = (OA)(OM) + (OA)(MP).$$

What is the meaning of this area equation? It is that the parallelogram $(OA)(OP)$ is equivalent to the product $(OA)(OM)$ together with the rectangle formed by OA and MP. This, in my opinion, is the fundamental principle of vector analysis (*Principles of the Algebra of Physics,* p. 72).

Let the vector OA be denoted by the black letter A, and the vector OP by the black letter R; let the rectangular co-ordinates of A be a,* b, c, and those of R be x, y, z, so that

$$\mathsf{A} = ai + bj + ck \text{ and } \mathsf{R} = xi + yj + zk.$$

Then the analytical product of the two vectors is

$$\begin{aligned}\mathsf{AR} &= (ai + bj + ck)(xi + yj + zk) \\ &= ax + by + cz + (bz - cy)jk + (cx - az)ki + (ay - bx)ij,\end{aligned}$$

and of the two partial products into which the complete product breaks up, the former, $ax + by + cz$, expresses $(OA)(OM)$, while the latter,

$$(bz - cy)jk + (cx - az)ki + (ay - bx)ij,$$

expresses $(OA)(MP)$.

It appears to me that the former partial product is correctly denoted by the expression cos AR, and the latter by the complementary expression Sin AR. The latter function is written with

---

[*] The letter a is in some places used to denote the magnitude of OA according to the usage of analysis; the context shows clearly whether it is the whole magnitude or the magnitude of the i-component which is meant.

a capital because it has an aspect or axis; it is not a simple area, but a directed area. The equation

$$(OA)(OP)=(OA)(OM)+(OA)(MP)$$

is then written

$$\mathsf{AR}=\cos\mathsf{AR}+\operatorname{Sin}\mathsf{AR}.$$

The notation sin AR serves for the magnitude of the sine product apart from its aspect or axis; it is the equivalent of the unwieldy Cartesian expression

$$\sqrt{(bz-cy)^2+(cx-az)^2+(ay-bx)^2}.$$

While $(OA)(MP)$ will be used to denote Sin AR; the notation $OA\times MP$ will be used to denote sin AR.

The function Sin AR cannot be expressed in rectangular coordinates without introducing symbols for the axes; hence it cannot be treated by the Cartesian analysis except indirectly.

Corresponding to the line equation

$$OP=OM+MP$$

there is the scalar equation

$$(OP)^2=(OM)^2+(MP)^2;$$

and corresponding to the area equation

$$\mathsf{AR}=\cos\mathsf{AR}+\operatorname{Sin}\mathsf{AR}$$

there is the scalar equation

$$\mathsf{A}^2\mathsf{R}^2=(\cos\mathsf{AR})^2+(\operatorname{Sin}\mathsf{AR})^2,$$

which, expanded in Cartesians, becomes

$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2+(bz-cy)^2$$
$$+(cx-az)^2+(ay-bx)^2.$$

If we take the vector which is the reciprocal of A, we get

$$\frac{1}{\mathsf{A}}\mathsf{R}=\frac{1}{OA}OM+\frac{1}{OA}MP$$
$$=\frac{OM}{OA}+\frac{1}{OA}MP.$$

When the order of the factors in a quotient is immaterial as in the cosine term, the quotient may be written in the ordinary way; when the order of the factors is essential as in the Sine term, the order will be indicated by introducing the reciprocal before or after according to the manner in which it enters. Hence by introducing OA in both numerator and denominator,

$$\frac{1}{\mathsf{A}}\mathsf{R} = \frac{(OA)(OM)}{(OA)^2} + \frac{(OA)(MP)}{(OA)^2}$$

$$= \frac{\mathsf{AR}}{\mathsf{A}^2}.$$

Hence $$\cos\frac{1}{\mathsf{A}}\mathsf{R} = \frac{OM}{OA} = \frac{(OA)(OM)}{(OA)^2} = \frac{ax+by+cz}{a^2+b^2+c^2} = \frac{\cos\mathsf{AR}}{\mathsf{A}^2},$$

and $$\operatorname{Sin}\frac{1}{\mathsf{A}}\mathsf{R} = \frac{1}{OA}MP = \frac{(OA)(MP)}{(OA)^2}$$

$$= \frac{(bz-cy)jk + (cx-az)ki + (ay-bx)ij}{a^2+b^2+c^2}$$

$$= \frac{\operatorname{Sin}\mathsf{AR}}{\mathsf{A}^2}.$$

Here no relation is imposed connecting A and R; their extremities are not restricted to lying on a circle or any other curve. Thus the functions are triangular or trigonometric in the primary sense of the word. We are introduced to the consideration of trigonometric areas as well as trigonometric lines and trigonometric ratios.

Second: we consider the lines OT and TA obtained by drawing AT at right angles to OA. As a line-vector equation we have

$$OA = OT + TA,$$

and from it we derive the area-vector equation

$$(OA)(OA) = (OA)(OT) + (OA)(TA),$$

or $$(OA)^2 = (OA)(OT) - (OA)(AT).$$

The latter equation means that the square of OA is in a certain sense equal to the difference between the parallelogram formed by OA and OT and the rectangle formed by OA and AT. In form it is merely a transformation of the area equation considered above (p. 13).

Let $(OA)(OT)$ be denoted by Sec AR and $(OA)(AT)$ by Tan AR, then the above equation is written

$$\mathsf{A}^2 = \mathrm{Sec}\,\mathsf{AR} - \mathrm{Tan}\,\mathsf{AR}.$$

Both functions are written with a capital, because each involves an aspect or axis.

After dividing by A^2 we obtain

$$1 = \frac{\mathrm{Sec}\,\mathsf{AR}}{\mathsf{A}^2} - \frac{\mathrm{Tan}\,\mathsf{AR}}{\mathsf{A}^2}$$

$$= \mathrm{Sec}\,\frac{1}{\mathsf{A}}\mathsf{R} - \mathrm{Tan}\,\frac{1}{\mathsf{A}}\mathsf{R}.$$

Corresponding to the line equation we have the scalar equation

$$(OA)^2 = (OT)^2 - (AT)^2,$$

and corresponding to the area equation we have the scalar equations

$$\mathsf{A}^4 = (\mathrm{Sec}\,\mathsf{AR})^2 - (\mathrm{Tan}\,\mathsf{AR})^2,$$

and
$$1 = \left(\mathrm{Sec}\,\frac{1}{\mathsf{A}}\mathsf{R}\right)^2 - \left(\mathrm{Tan}\,\frac{1}{\mathsf{A}}\mathsf{R}\right)^2.$$

To find the expressions for these trigonometrical functions in terms of rectangular co-ordinates, we proceed as follows. Since

$$OT = \frac{OA}{OM} OP$$

and
$$AT = \frac{OA}{OM} MP;$$

therefore

$$(OA)^2 = \frac{OA}{OM}(OA)(OP) - \frac{OA}{OM}(OA)(MP)$$

$$= \frac{(OA)^2}{(OM)(OA)}(OA)(OP) - \frac{(OA)^2}{(OM)(OA)}(OA)(MP);$$

that is, $\mathsf{A}^2 = \frac{\mathsf{A}^2}{\cos \mathsf{AR}}\mathsf{AR} - \frac{\mathsf{A}^2}{\cos \mathsf{AR}}\mathrm{Sin}\,\mathsf{AR}.$

Hence $$\mathrm{Sec}\,\mathsf{AR} = \frac{\mathsf{A}^2}{\cos \mathsf{AR}}\,\mathsf{AR}$$

$$= \frac{a^2+b^2+c^2}{ax+by+cz}(ai+bj+ck)\,(xi+yj+zk),$$

and $$\mathrm{Tan}\,\mathsf{AR} = \frac{\mathsf{A}^2}{\cos \mathsf{AR}}\,\mathrm{Sin}\,\mathsf{AR}$$

$$= \frac{a^2+b^2+c^2}{ax+by+cz}\{(bz-cy)jk+(cx-az)ki+(ay-bx)ij\}.$$

Hence $$\mathrm{Sec}\,\frac{1}{\mathsf{A}}\,\mathsf{R} = \frac{\mathsf{AR}}{\cos \mathsf{AR}} = \frac{(ai+bj+ck)\,(xi+yj+ck)}{ax+by+cz},$$

and $$\mathrm{Tan}\,\frac{1}{\mathsf{A}}\,\mathsf{R} = \frac{\mathrm{Sin}\,\mathsf{AR}}{\cos \mathsf{AR}} = \frac{(bz-cy)jk+(cx-az)ki+(ay-bx)ij}{ax+by+cz}.$$

The function sec AR is obtained from Sec AR by substituting the appropriate square roots of $(ai+bj+ck)^2$ and $(xi+yj+zk)^2$. Similarly, the function tan AR is obtained from Tan AR by substituting the appropriate square root of (Sin AR)^2. By sec AR is meant the magnitude of Sec AR, and by tan AR the magnitude of Tan AR.

Third: we consider the lines OQ and QP obtained by drawing PQ at right angles to OP. We have the line-vector equation

$$OP = OQ + QP$$

with the corresponding scalar equation

$$(OP)^2 = (OQ)^2 - (QP)^2.$$

From the former we derive the area-vector equation

$$(OA)\,(OP) = (OA)\,(OQ) + (OA)\,(QP),$$

which means that the parallelogram OA, OP is in a certain sense equivalent to the product of the two codirectional lines OA and OQ together with the parallelogram OA, QP. The two parallelograms are on the same base and between the same parallels, but the angle of the latter exceeds the angle of the former by a quadrant. For the sake of clearness it is absolutely necessary to devise a distinctive notation for the products in question. As the line PQ is drawn from OP in the same manner as AT from

OA, the line OQ partakes of the nature of the Sec line OT, and the line QP partakes of the nature of the Tan line AT. By changing the initial consonants from light to heavy, we obtain a notation which is suggestive and easily remembered, and will serve at least for the purpose of this investigation.

Let, then, $(OA)(OQ)$ be denoted by zec AR, and $(OA)(QP)$ by Dan AR; the above equation is then written

$$\mathsf{AR} = \text{zec}\,\mathsf{AR} + \text{Dan}\,\mathsf{AR}.$$

As
$$\frac{\mathsf{AR}}{\mathsf{A}^2} = \frac{\text{zec}\,\mathsf{AR}}{\mathsf{A}^2} + \frac{\text{Dan}\,\mathsf{AR}}{\mathsf{A}^2},$$

therefore
$$\frac{1}{\mathsf{A}}\mathsf{R} = \text{zec}\frac{1}{\mathsf{A}}\mathsf{R} + \text{Dan}\frac{1}{\mathsf{A}}\mathsf{R}.$$

The corresponding scalar equations are

$$\mathsf{A}^2\mathsf{R}^2 = (\text{zec}\,\mathsf{AR})^2 - (\text{Dan}\,\mathsf{AR})^2,$$

and
$$\frac{\mathsf{R}^2}{\mathsf{A}^2} = \left(\text{zec}\frac{1}{\mathsf{A}}\mathsf{R}\right)^2 - \left(\text{Dan}\frac{1}{\mathsf{A}}\mathsf{R}\right)^2.$$

To find the expressions for these functions in terms of rectangular co-ordinates, we proceed as follows:

Since $OQ = \frac{(OP)^2}{OM}$, and $QP = \frac{MP}{OM}\sqrt{-1}\,OP$, where $\sqrt{-1}\,OP$ denotes that the line OP is turned through a positive quadrant in the given plane; we deduce that

$$(OA)(OP) = \frac{(OA)(OP)^2}{OM} + \frac{MP}{OM}(OA)(\sqrt{-1}\,OP)$$

$$= \frac{(OA)^2(OP)^2}{(OA)(OM)} + \frac{(OA)(MP)}{(OA)(OM)}(OA)(\sqrt{-1}\,OP),$$

therefore
$$\mathsf{AR} = \frac{\mathsf{A}^2\mathsf{R}^2}{\cos\mathsf{AR}} + \frac{\sin\mathsf{AR}}{\cos\mathsf{AR}}\mathsf{A}\sqrt{-1}\,\mathsf{R}.$$

Hence $\text{zec}\,\mathsf{AR} = \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax+by+cz}$,

and Dan $\mathsf{AR} =$

$$\frac{\sqrt{(bz-cy)^2+(cx-az)^2+(ay-bx)^2}}{ax+by+cz}(ai+bj+ck)\sqrt{-1}(xi+yj+zk).$$

Similarly $$\operatorname{zec}\frac{1}{\mathsf{A}}\mathsf{R} = \frac{x^2+y^2+z^2}{ax+by+cz},$$

while $\operatorname{Dan}\frac{1}{\mathsf{A}}\mathsf{R}$ is obtained by dividing $\operatorname{Dan}\mathsf{AR}$ by $a^2+b^2+c^2$,

and $\operatorname{dan}\mathsf{AR} =$

$$\frac{\sqrt{(bz-cy)^2+(cx-az)^2+(ay-bx)^2}\sqrt{a^2+b^2+c^2}\sqrt{x^2+y^2+z^2}}{ax+by+cz}.$$

Thus $\operatorname{zec}\mathsf{AR}$ is the reciprocal of $\cos\mathsf{AR}$, not with respect to unity, but with respect to $\mathsf{A}^2\mathsf{R}^2$; while $\operatorname{zec}\frac{1}{\mathsf{A}}\mathsf{R}$ is the reciprocal of $\cos\frac{1}{\mathsf{A}}\mathsf{R}$ with respect to $\frac{\mathsf{R}^2}{\mathsf{A}^2}$.

Fourth: we consider the lines OS and SA obtained by drawing AS perpendicular to OP.

We have the line-vector equation

$$OA = OS + SA,$$

and from it we derive the area-vector equation

$$\begin{aligned}(OA)^2 &= (OA)(OS) + (OA)(SA)\\ &= (OA)(OS) - (OA)(AS).\end{aligned}$$

This equation means that the square of OA is equal to the difference between the parallelograms OA, OS and OA, AS. As the lines OS and AS have a certain analogy to the lines OM and MP, let the products be denominated by Gos and Zin, the initial consonants of the functions being changed from light to heavy. The above equation is then written

$$A^2 = \operatorname{Gos}\mathsf{AR} - \operatorname{Zin}\mathsf{AR}.$$

Since $$OS = \frac{OA}{OQ}OP = \frac{(OM)(OA)}{OP^2}OP = \frac{\cos\mathsf{AR}}{\mathsf{R}^2}OP,$$

and $$AS = \frac{OA}{OQ}QP = \frac{(OA)(MP)}{(OP)^2}\sqrt{-1}\,OP = \frac{\sin\mathsf{AR}}{\mathsf{R}^2}\sqrt{-1}\,OP,$$

the above equation becomes

$$\mathsf{A}^2 = \frac{\cos\mathsf{AR}}{\mathsf{R}^2}\mathsf{AR} - \frac{\sin\mathsf{AR}}{\mathsf{R}^2}\mathsf{A}\sqrt{-1}\,\mathsf{R}.$$

Hence $\mathrm{Gos}\,\mathsf{AR} = \dfrac{ax+by+cz}{x^2+y^2+z^2}(ai+bj+ck)(xi+yj+zk)$,

and $$\mathrm{gos}\,\mathsf{AR} = (ax+by+cz)\frac{\sqrt{a^2+b^2+c^2}}{\sqrt{x^2+y^2+z^2}}.$$

The *versed-sine* product is obtained by considering AP the third side of the triangle. Because

$$AP = AO + OP,$$

therefore
$$\begin{aligned}(OA)(AP) &= (OA)(AO)+(OA)(OP)\\ &= -(OA)^2+(OA)(OP).\end{aligned}$$

Hence $$\cos(OA)(AP) = -(OA)^2+\cos(OA)(OP),$$

and $$\mathrm{Sin}(OA)(AP) = \mathrm{Sin}\,(OA)(OP).$$

It is the new product $\cos(OA)(AP)$ which is properly called vers AR; so that
$$\begin{aligned}\mathrm{vers}\,\mathsf{AR} &= -\mathsf{A}^2+\cos\mathsf{AR}\\ &= (OA)(AM).\end{aligned}$$

Similarly
$$\begin{aligned}\mathrm{vers}\,\frac{1}{\mathsf{A}}\mathsf{R} &= -1+\frac{\cos\mathsf{AR}}{\mathsf{A}^2}\\ &= \frac{AM}{OA}.\end{aligned}$$

According to this definition the versine is negative when the point M falls to the left of A; for OA and AM then have opposite directions. In circular trigonometry it is commonly stated that the versine is always positive; it is more correct to say that in the case of the circular functions the versine is always negative.

Finally, we have to consider the definitions of the complementary functions. By the *complementary-vector* of A with respect to R is meant the vector OB (Fig. 6), which is equal and perpendicular to A in the plane of A and R, and drawn to the side of A on which R is (*Principles of the Algebra of Physics*, p. 87). Let it be denoted by $\overline{\mathsf{A}}$, the horizontal bar denoting "perpendicular to." When all the lines lie in a common plane, this notation is definite. Grassmann uses a vertical bar prefixed to the vector it refers to, as $|\mathsf{A}$. The horizontal bar is preferable, because in space it must be attached to a pair of vectors, and the horizontal form allows this to be done conveniently. The complementary vector is expressed in terms of A and R by the equation

$$\overline{\mathsf{A}} = \frac{\mathrm{Sin}(\mathrm{Sin}\,\mathsf{AR})\,\mathsf{A}}{\sin \mathsf{AR}}$$

where

$$\begin{aligned}\mathrm{Sin}(\mathrm{Sin}\,\mathsf{AR})\,\mathsf{A} &= \{(cx-az)c-(ay-bx)b\}i \\ &+\{(ay-bx)a-(bz-cy)c\}j \\ &+\{(bz-cy)b-(cx-az)a\}k.\end{aligned}$$

By the complementary function is meant the function which is obtained when $\overline{\mathsf{A}}$ is substituted for A in the original function. Draw PN perpendicular to OB, and PU to OP; BV perpendicular to OB, and BW to OP. The prefix *co-* may be used to denote the complementary function. The geometrical definitions then are

$$\begin{aligned}
\text{co-cos}\,\mathsf{AR} &= (OB)(ON), & \text{co-Sin}\,\mathsf{AR} &= (OB)(NP), \\
\text{co-Sec}\,\mathsf{AR} &= (OB)(OV), & \text{co-Tan}\,\mathsf{AR} &= (OB)(BV), \\
\text{co-zec}\,\mathsf{AR} &= (OB)(OU), & \text{co-Dan}\,\mathsf{AR} &= (OB)(UP), \\
\text{co-Gos}\,\mathsf{AR} &= (OB)(OW), & \text{co-Zin}\,\mathsf{AR} &= (OB)(BW).
\end{aligned}$$

It may be shown that co-cos $\mathsf{AR} = \sin \mathsf{AR}$. Also co-Sin AR may be denoted by Cos AR; it is equal to $-\frac{\cos \mathsf{AR}}{\sin \mathsf{AR}}\mathrm{Sin}\,\mathsf{AR}$.

The several trigonometric areas are exhibited synoptically in the following table. It is evident that Hamilton's S and V are entirely inadequate to express the various scalar and vector functions of the product of two vectors.

TRIGONOMETRIC AREAS.

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
AR	$(OA)(OP)$	$(ai + bj + ck)(xi + yj + zk)$
cos AR	$(OA)(OM)$	$ax + by + cz$
Sin AR	$(OA)(MP)$	$(bz - cy)jk + (cx - az)ki + (ay - bx)ij$
sin AR	$OA \times MP$	$\sqrt{(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2}$
Sec AR	$(OA)(OT)$	$\frac{\mathrm{A}^2}{\cos \mathrm{AR}} \mathrm{AR}$
sec AR	$OA \times OT$	$\frac{\mathrm{A}^2}{\cos \mathrm{AR}} \sqrt{\mathrm{A}^2\mathrm{R}^2}$
Tan AR	$(OA)(AT)$	$\frac{\mathrm{A}^2}{\cos \mathrm{AR}} \mathrm{Sin\ AR}$
tan AR	$OA \times AT$	$\frac{\mathrm{A}^2}{\cos \mathrm{AR}} \sin \mathrm{AR}$
zec AR	$(OA)(OQ)$	$\frac{\mathrm{A}^2\mathrm{R}^2}{\cos \mathrm{AR}}$
Dan AR	$(OA)(QP)$	$\frac{\sin \mathrm{AR}}{\cos \mathrm{AR}} \mathrm{A}\sqrt{-1}\,\mathrm{R}$
dan AR	$OA \times QP$	$\frac{\sin \mathrm{AR}}{\cos \mathrm{AR}} \sqrt{\mathrm{A}^2\mathrm{R}^2}$
Gos AR	$(OA)(OS)$	$\frac{\cos \mathrm{AR}}{\mathrm{R}^2} \mathrm{AR}$
gos AR	$OA \times OS$	$\frac{\cos \mathrm{AR}}{\mathrm{R}^2} \sqrt{\mathrm{A}^2\mathrm{R}^2}$
Zin AR	$(OA)(AS)$	$\frac{\sin \mathrm{AR}}{\mathrm{R}^2} \mathrm{A}\sqrt{-1}\,\mathrm{R}$
zin AR	$OA \times AS$	$\frac{\sin \mathrm{AR}}{\mathrm{R}^2} \sqrt{\mathrm{A}^2\mathrm{R}^2}$
vers AR	$(OA)(AM)$	$-\mathrm{A}^2 + \cos \mathrm{AR}$
co-cos AR	$(OB)(ON)$	$\sin \mathrm{AR}$

TRIGONOMETRIC AREAS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
co-Sin AR	$(OB)(NP)$	$-\frac{\cos \mathrm{AR}}{\sin \mathrm{AR}} \mathrm{Sin\ AR} = \mathrm{Cos\ AR}$
co-sin AR	$OB \times NP$	$\cos \mathrm{AR}$
co-Sec AR	$(OB)(OV)$	$\frac{\mathrm{A}^2}{\sin \mathrm{AR}} \overline{\mathrm{A}}\mathrm{R}$
co-sec AR	$OB \times OV$	$\frac{\mathrm{A}^2}{\sin \mathrm{AR}} \sqrt{\mathrm{A}^2\mathrm{R}^2}$
co-Tan AR	$(OB)(BV)$	$\frac{\mathrm{A}^2}{\sin \mathrm{AR}} \mathrm{Cos\ AR}$
co-tan AR	$OB \times BV$	$\mathrm{A}^2 \frac{\cos \mathrm{AR}}{\sin \mathrm{AR}}$
co-zec AR	$(OB)(OU)$	$\frac{\mathrm{A}^2\mathrm{R}^2}{\sin \mathrm{AR}}$
co-Dan AR	$(OB)(UP)$	$\frac{\cos \mathrm{AR}}{\sin \mathrm{AR}} \overline{\mathrm{A}}\sqrt{-1}\ \mathrm{R}$
co-dan AR	$OB \times UP$	$\frac{\cos \mathrm{AR}}{\sin \mathrm{AR}} \sqrt{\mathrm{A}^2\mathrm{R}^2}$
co-Gos AR	$(OB)(OW)$	$\frac{\sin \mathrm{AR}}{\mathrm{R}^2} \overline{\mathrm{A}}\mathrm{R}$
co-gos AR	$OB \times OW$	$\frac{\sin \mathrm{AR}}{\mathrm{R}^2} \sqrt{\mathrm{A}^2\mathrm{R}^2}$
co-Zin AR	$(OB)(BW)$	$\frac{\cos \mathrm{AR}}{\mathrm{R}^2} \overline{\mathrm{A}}\sqrt{-1}\ \mathrm{R}$
co-zin AR	$OB \times BW$	$\frac{\cos \mathrm{AR}}{\mathrm{R}^2} \sqrt{\mathrm{A}^2\mathrm{R}^2}$
co-vers AR	$(OB)(BN)$	$-\mathrm{A}^2 + \sin \mathrm{AR}$

THE CIRCULAR FUNCTIONS.

In the case of the circular functions the variable vector R is always of the same length as the initial vector A; in other words, *OP* is limited by the condition that its extremity must lie on a circle of radius *OA* (Fig. 7). There is a definite area enclosed between *OA*, *OP* and the arc *AP*; and the triangular functions can be expressed as functions of this area. Let *A* denote the area of the sector *AOP*, s the length of the arc *AP*, and a the magnitude of *OA*; then $\frac{2A}{a^2}=\frac{s}{a}$. Let this quantity be denoted by u; it is the circular measure of the angle *AOP*, and is more properly regarded as the ratio of twice the area of the sector *AOP* to the square on *OA* than as the ratio of the arc *AP* to the line *OA*.

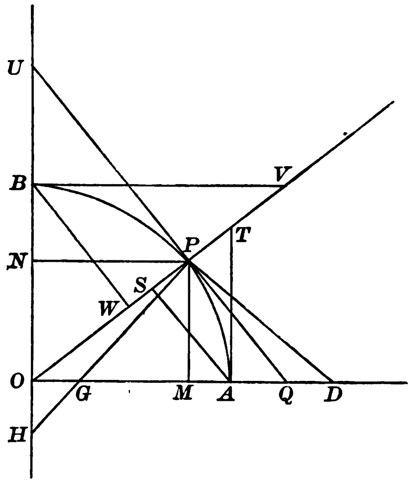

FIG. 7.

The following table shows that the circular ratio is deduced from the corresponding trigonometric area by dividing by A^2, and introducing the special relation that

$$(\cos AR)^2 + (\operatorname{Sin} AR)^2 = A^4,$$

or

$$R^2 = A^2.$$

In addition to the triangular lines there are the curve lines or circular lines proper; namely, the tangent, the secant, the normal, etc. By the *tangent* is meant the line DP drawn from a point in OA so as to touch the curve at P, and by the *secant* is meant the line OD cut off. By the *normal* is meant the line GP which starts from the line OA, and is at right angles to the tangent at P, while OG is the complementary line. Let these functions be denoted by Tnt, Sct, Nor, respectively.

Since $$DP = DM + MP,$$

$$(OA)(DP) = (OA)(DM) + (OA)(MP)$$
$$= (OA)(DM) + \text{Sin AR}.$$

But, generally, $$DM = \sin\frac{d(\cos)}{d(\sin)}OA,$$

which, for the special case of the circle, becomes

$$DM = -\frac{\sin^2 u}{\cos u}OA,$$

therefore $$(OA)(DP) = -\frac{(\text{Sin AR})^2}{\cos \text{AR}} + \text{Sin AR}.$$

Again, $$(OA)(OD) = (OA)(OM) + (OA)(MD)$$
$$= \cos \text{AR} + (OA)(MD),$$

which, for the case of the circle, becomes

$$(OA)(OD) = \cos \text{AR} + \frac{(\text{Sin AR})^2}{\cos \text{AR}}$$
$$= \frac{\text{A}^2\text{R}^2}{\cos \text{AR}}$$
$$= \frac{\text{A}^4}{\cos \text{AR}}.$$

For the normal we have the general relation

$$GP = GM + MP,$$

therefore $$(OA)(GP) = (OA)(GM) + (OA)(MP)$$
$$= (OA)(GM) + \text{Sin AR}$$
$$= -\sin\frac{d(\sin)}{d(\cos)}(OA)^2 + \text{Sin AR}.$$

Hence for the special case of the circle

$$\begin{aligned}(OA)(GP) &= \cos \mathsf{AR} + \operatorname{Sin} \mathsf{AR} \\ &= \mathsf{AR};\end{aligned}$$

hence GP is identical with OP.

Finally,
$$\begin{aligned}OG &= OM + MG, \\ (OA)(OG) &= (OA)(OM) + (OA)(MG) \\ &= \cos \mathsf{AR} + \sin \frac{d(\sin)}{d(\cos)}(OA)^2,\end{aligned}$$

therefore for the special case of the circle

$$\begin{aligned}(OA)(OG) &= \cos \mathsf{AR} - \cos \mathsf{AR} \\ &= 0.\end{aligned}$$

The ratios are defined by taking the ratio of the corresponding area to A^2; thus

$$\operatorname{sct} u = \frac{OD}{OA} = \frac{\mathsf{A}^2}{\cos \mathsf{AR}} = \operatorname{zec} u,$$

$$\operatorname{Tnt} u = \frac{1}{OA} DP = -\frac{(\operatorname{Sin} \mathsf{AR})^2}{\mathsf{A}^2 \cos \mathsf{AR}} + \frac{\operatorname{Sin} \mathsf{AR}}{\mathsf{A}^2} = \operatorname{Dan} u,$$

$$\operatorname{tnt} u = \frac{DP}{OA} = \frac{\sin \mathsf{AR}}{\cos \mathsf{AR}} = \tan u = \operatorname{dan} u,$$

$$\operatorname{Nor} u = \frac{1}{OA} GP = \frac{\mathsf{AR}}{\mathsf{A}^2},$$

$$\operatorname{nor} u = \frac{GP}{OA} = 1,$$

$$\operatorname{anon} u = \frac{OG}{OA} = 0.$$

Answering to each curve-ratio there is a complementary curve-ratio. In Fig. 7 EP is the co-tangent line, and HP represents the co-normal line. For the circle, E coincides with U. Then

$$\text{co-sct}\, u = \frac{OE}{OB}, \quad \text{co-Tnt}\, u = \frac{1}{OB} EP, \quad \text{co-tnt}\, u = \frac{EP}{OB},$$

$$\text{co-Nor}\, u = \frac{1}{OB} HP, \quad \text{co-nor}\, u = \frac{HP}{OB}.$$

CIRCULAR RATIOS.

Function.	Geometric Definition.	Analytical Definition.
α^u	$\frac{1}{OA}OP$	$\frac{AR}{A^2}$
cos u	$\frac{OM}{OA}$	$\frac{\cos AR}{A^2}$
Sin u	$\frac{1}{OA}MP$	$\frac{\text{Sin}\,AR}{A^2}$
sin u	$\frac{MP}{OA}$	$\frac{\sin AR}{A^2}$
Sec u	$\frac{1}{OA}OT$	$\frac{AR}{\cos AR}$
sec u	$\frac{OT}{OA}$	$\frac{A^2}{\cos AR}$
Tan u	$\frac{1}{OA}AT$	$\frac{\text{Sin}\,AR}{\cos AR}$
tan u	$\frac{AT}{OA}$	$\frac{\sin AR}{\cos AR}$
zec u	$\frac{OQ}{OA}$	$\frac{A^2}{\cos AR}$
Dan u	$\frac{1}{OA}QP$	$\frac{1}{A^2}\frac{\sin AR}{\cos AR}A\sqrt{-1}\,R$
dan u	$\frac{QP}{OA}$	$\frac{\sin AR}{\cos AR}$
Gos u	$\frac{1}{OA}OS$	$\frac{\cos AR}{A^4}AR$
gos u	$\frac{OS}{OA}$	$\frac{\cos AR}{A^2}$
Zin u	$\frac{1}{OA}AS$	$\frac{\sin AR}{A^4}A\sqrt{-1}\,R$
zin u	$\frac{AS}{OA}$	$\frac{\sin AR}{A^2}$
vers u	$\frac{AM}{OA}$	$-1+\frac{\cos AR}{A^2}$
sct u	$\frac{OD}{OA}$	$\frac{A^2}{\cos AR}$

CIRCULAR RATIOS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
Tnt *u*	$\frac{1}{OA}DP$	$-\frac{(\operatorname{Sin} AR)^2}{A^2 \cos AR}+\frac{\operatorname{Sin} AR}{A^2}$
tnt *u*	$\frac{DP}{OA}$	$\frac{\sin AR}{\cos AR}$
Nor *u*	$\frac{1}{OA}GP$	$\frac{AR}{A^2}$
nor *u*	$\frac{GP}{OA}$	1
co-cos *u*	$\frac{ON}{OB}$	$\frac{\sin AR}{A^2}$
co-Sin *u*	$\frac{1}{OB}NP$	$\frac{\operatorname{Cos} AR}{A^2}$
co-sin *u*	$\frac{NP}{OB}$	$\frac{\cos AR}{A^2}$
co-Sec *u*	$\frac{1}{OB}OV$	$\frac{\bar{A}R}{\sin AR}$
co-sec *u*	$\frac{OV}{OB}$	$\frac{A^2}{\sin AR}$
co-Tan *u*	$\frac{1}{OB}BV$	$\frac{\operatorname{Cos} AR}{\sin AR}$
co-tan *u*	$\frac{BV}{OB}$	$\frac{\cos AR}{\sin AR}$
co-zec *u*	$\frac{OU}{OB}$	$\frac{A^2}{\sin AR}$
co-Dan *u*	$\frac{1}{OB}UP$	$\frac{\cos AR}{\sin AR}\ \frac{\bar{A}\sqrt{-1}\,R}{A^2}$
co-dan *u*	$\frac{UP}{OB}$	$\frac{\cos AR}{\sin AR}$
co-Gos *u*	$\frac{1}{OB}OW$	$\frac{\sin AR}{A^2}\ \frac{\bar{A}R}{A^2}$
co-gos *u*	$\frac{OW}{OB}$	$\frac{\sin AR}{A^2}$
co-Zin *u*	$\frac{1}{OB}BW$	$\frac{\cos AR}{A^2}\ \frac{\bar{A}\sqrt{-1}\,R}{A^2}$

CIRCULAR RATIOS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
co-zin u	$\frac{BW}{OB}$	$\frac{\cos \mathsf{AR}}{\mathsf{A}^2}$
co-sct u	$\frac{OE}{OB}$	$\frac{\mathsf{A}^2}{\sin \mathsf{AR}}$
co-Tnt u	$\frac{1}{OB} EP$	$-\frac{(\mathrm{Cos}\, \mathsf{AR})^2}{\mathsf{A}^2 \sin \mathsf{AR}} + \frac{\mathrm{Cos}\, \mathsf{AR}}{\mathsf{A}^2}$
co-tnt u	$\frac{EP}{OB}$	$\frac{\cos \mathsf{AR}}{\sin \mathsf{AR}}$
co-Nor u	$\frac{1}{OB} HP$	$\frac{\bar{\mathsf{A}}\mathsf{R}}{\mathsf{A}^2}$
co-nor u	$\frac{HP}{OB}$	1

As a test of the accuracy of these definitions, let us consider how they apply to the proof of the addition theorem for two circular sectors having a common plane. Let AOP and POQ be the successive coplanar sectors (Fig. 8); PM and QK are drawn perpendicular to OA, QN is drawn perpendicular to OP, and from the point N so determined NL is drawn perpendicular to OA, and NR perpendicular to QK. By definition,

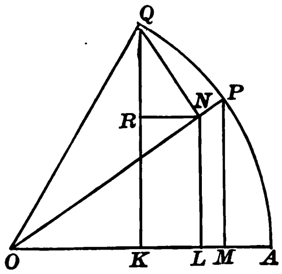

FIG. 8.

$$\cos u = \frac{OM}{OA}, \qquad \sin u = \frac{MP}{OA},$$

$$\cos v = \frac{ON}{OP}, \qquad \sin v = \frac{NQ}{OP},$$

and $$\cos(u+v) = \frac{OK}{OA}, \qquad \sin(u+v) = \frac{KQ}{OA}.$$

Now
$$\cos(u+v)=\frac{OK}{OA}$$
$$=\frac{OL}{OA}+\frac{LK}{OA},$$

and $OL=\frac{ON}{OP}OM$ on account of the similarity of the triangles LON and MOP,

and
$$LK=NR=\frac{MP}{OP}QN,$$

on account of the similarity of the triangles MOP and RQN, and the negative nature of NR with respect to OA;

therefore
$$\cos(u+v)=\frac{ON}{OP}\frac{OM}{OA}+\frac{MP}{OP}\frac{QN}{OA}$$
$$=\frac{OM}{OA}\frac{ON}{OP}-\frac{MP}{OA}\frac{NQ}{OP}$$
$$=\cos u\cos v-\sin u\sin v.$$

In a similar manner
$$\sin(u+v)=\frac{KQ}{OA}$$
$$=\frac{LN}{OA}+\frac{RQ}{OA}$$
$$=\frac{ON}{OA}\frac{MP}{OP}+\frac{NQ}{OP}\frac{OM}{OA}$$
$$=\frac{MP}{OA}\frac{ON}{OP}+\frac{OM}{OA}\frac{NQ}{OP}$$
$$=\sin u\cos v+\cos u\sin v.$$

THE EXCIRCULAR FUNCTIONS.

In this case the bounding line AP (Fig. 9) is part of a rectangular hyperbola or excircle, having OA for principal axis. Let s denote the length of the arc AP, a the length of OA, and A the area of AOP; the analogue of the circular u is no longer

$\frac{s}{a}$, but it still is $\frac{2A}{a^2}$. All the triangular ideas and all the curve ideas which apply to the circle apply also to the excircle, and they are expressed by analogous functions of u. These functions are appropriately denominated by the same names, while for distinction the qualification "hyperbolic" is introduced. The abbreviations for the functions are distinguished by an appended h.

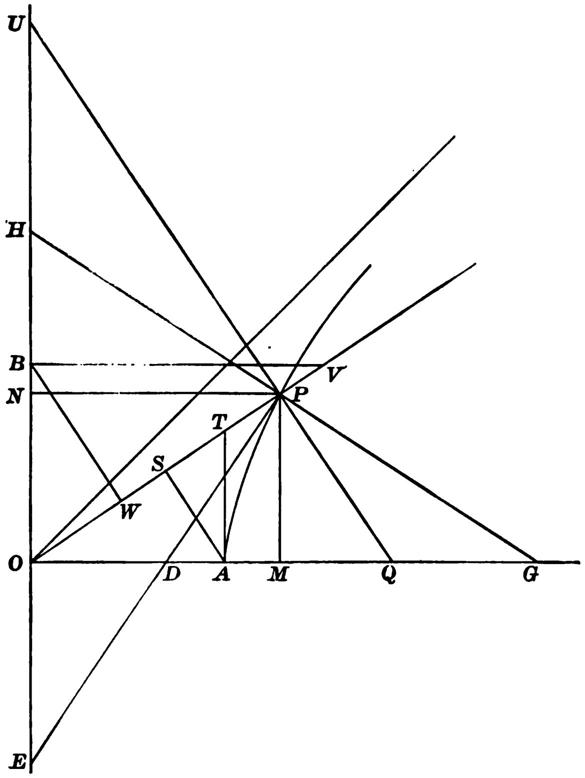

FIG. 9.

The analytical definition is obtained by dividing the corresponding area function by A², and adding the condition that

$$(\cos \mathrm{AR})^2 - (\mathrm{Sin}\ \mathrm{AR})^2 = \mathrm{A}^4.$$

In the case of the excircle

$$DM = \sinh \frac{d(\cosh)}{d(\sinh)} OA$$

$$= \frac{(\sinh u)^2}{\cosh u} OA$$

$$= \frac{(\text{Sin AR})^2}{a \cos \text{AR}}.$$

Consequently, $$(OA)(DP) = \frac{(\text{Sin AR})^2}{\cos \text{AR}} + \text{Sin AR},$$

and $$(OA)(OD) = \cos \text{AR} - \frac{(\text{Sin AR})^2}{\cos \text{AR}}$$

$$= \frac{\text{A}^4}{\cos \text{AR}}.$$

Again, for the excircle

$$GM = -\sinh \frac{d(\sinh)}{d(\cosh)} OA$$

$$= -\cosh u \, OA$$

$$= -\frac{\cos \text{AR}}{a};$$

consequently, $$(OA)(GP) = -\cos \text{AR} + \text{Sin AR}.$$

Hence GP is the reflection of OP with respect to MP,

and $$(OA)(OG) = 2 \cos \text{AR}.$$

When the radius-vector is subject to the hyperbolic condition, the several lines drawn according to their definitions are all different from one another; from which we see the necessity for these exact definitions.

EXCIRCULAR RATIOS.

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
α^{iu}	$\frac{1}{OA}OP$	$\frac{AR}{A^2}$
cosh u	$\frac{OM}{OA}$	$\frac{\cos AR}{A^2}$
Sinh u	$\frac{1}{OA}MP$	$\frac{\text{Sin}\ AR}{A^2}$
sinh u	$\frac{MP}{OA}$	$\frac{\sin AR}{A^2}$
Sech u	$\frac{1}{OA}OT$	$\frac{AR}{\cos AR}$
sech u	$\frac{OT}{OA}$	$\frac{\sqrt{A^2R^2}}{\cos AR}$
Tanh u	$\frac{1}{OA}AT$	$\frac{\text{Sin}\ AR}{\cos AR}$
tanh u	$\frac{AT}{OA}$	$\frac{\sin AR}{\cos AR}$
zech u	$\frac{OQ}{OA}$	$\frac{R^2}{\cos AR}$
Danh u	$\frac{1}{OA}QP$	$\frac{\sin AR}{\cos AR}\ \frac{A\sqrt{-1}\,R}{A^2}$
danh u	$\frac{QP}{OA}$	$\frac{\sin AR}{\cos AR}\ \frac{\sqrt{A^2R^2}}{A^2}$
Gosh u	$\frac{1}{OA}OS$	$\frac{\cos AR}{A^2R^2}\ AR$
gosh u	$\frac{OS}{OA}$	$\frac{\cos AR}{\sqrt{A^2R^2}}$
Zinh u	$\frac{1}{OA}AS$	$\frac{\sin AR}{A^2R^2}\ A\sqrt{-1}\,R$
zinh u	$\frac{AS}{OA}$	$\frac{\sin AR}{\sqrt{A^2R^2}}$
versh u	$\frac{AM}{OA}$	$-1+\frac{\cos AR}{A^2}$
scth u	$\frac{OD}{OA}$	$\frac{A^2}{\cos AR}$

EXCIRCULAR RATIOS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
Tnth *u*	$\frac{1}{OA}DP$	$\frac{(\text{Sin AR})^2}{\text{A}^2\cos\text{AR}}+\frac{\text{Sin AR}}{\text{A}^2}$
tnth *u*	$\frac{DP}{OA}$	$\frac{\sin\text{AR}}{\cos\text{AR}}\;\frac{\sqrt{\text{A}^2\text{R}^2}}{\text{A}^2}$
Norh *u*	$\frac{1}{OA}GP$	$-\frac{\cos\text{AR}}{\text{A}^2}+\frac{\text{Sin AR}}{\text{A}^2}$
norh *u*	$\frac{GP}{OA}$	$\frac{\sqrt{\text{A}^2\text{R}^2}}{\text{A}^2}$
co-cosh *u*	$\frac{ON}{OB}$	$\frac{\sin\text{AR}}{\text{A}^2}$
co-Sinh *u*	$\frac{1}{OB}NP$	$\frac{\text{Cos AR}}{\text{A}^2}$
co-sinh *u*	$\frac{NP}{OB}$	$\frac{\cos\text{AR}}{\text{A}^2}$
co-Sech *u*	$\frac{1}{OB}OV$	$\frac{\bar{\text{A}}\text{R}}{\sin\text{AR}}$
co-sech *u*	$\frac{OV}{OB}$	$\frac{\sqrt{\text{A}^2\text{R}^2}}{\sin\text{AR}}$
co-Tanh u	$\frac{1}{OB}BV$	$\frac{\text{Cos AR}}{\sin\text{AR}}$
co-tanh *u*	$\frac{BV}{OB}$	$\frac{\cos\text{AR}}{\sin\text{AR}}$
co-zech *u*	$\frac{OU}{OB}$	$\frac{\text{R}^2}{\sin\text{AR}}$
co-Danh *u*	$\frac{1}{OB}UP$	$\frac{\cos\text{AR}}{\sin\text{AR}}\;\frac{\bar{\text{A}}\sqrt{-1}\,\text{R}}{\text{A}^2}$
co-danh *u*	$\frac{UP}{OB}$	$\frac{\cos\text{AR}}{\sin\text{AR}}\;\frac{\sqrt{\text{A}^2\text{R}^2}}{\text{A}^2}$
co-Gosh *u*	$\frac{1}{OB}OW$	$\frac{\sin\text{AR}}{\text{A}^2\text{R}^2}\bar{\text{A}}\text{R}$
co-gosh *u*	$\frac{OW}{OB}$	$\frac{\sin\text{AR}}{\sqrt{\text{A}^2\text{R}^2}}$
co-Zinh *u*	$\frac{1}{OB}BW$	$\frac{\cos\text{AR}}{\text{A}^2\text{R}^2}\bar{\text{A}}\sqrt{-1}\,\text{R}$

EXCIRCULAR RATIOS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
co-zinh u	$\frac{BW}{OB}$	$\frac{\cos \mathrm{AR}}{\sqrt{\mathrm{A}^2\mathrm{R}^2}}$
co-versh u	$\frac{BN}{OB}$	$-1+\frac{\sin \mathrm{AR}}{\mathrm{A}^2}$
co-scth u	$\frac{OE}{OB}$	$-\frac{\mathrm{A}^2}{\sin \mathrm{AR}}$
co-Tnth u	$\frac{1}{OB}EP$	$\frac{(\mathrm{Cos\ AR})^2}{\mathrm{A}^2 \sin \mathrm{AR}}+\frac{\mathrm{Cos\ AR}}{\mathrm{A}^2}$
co-tnth u	$\frac{EP}{OB}$	$\frac{\cos \mathrm{AR}}{\sin \mathrm{AR}} \quad \frac{\sqrt{\mathrm{A}^2\mathrm{R}^2}}{\mathrm{A}^2}$
co-Norh u	$\frac{1}{OB}HP$	$-\frac{\sin \mathrm{AR}}{\mathrm{A}^2}+\frac{\mathrm{Cos\ AR}}{\mathrm{A}^2}$
co-norh u	$\frac{HP}{OB}$	$\frac{\sqrt{\mathrm{A}^2\mathrm{R}^2}}{\mathrm{A}^2}$

Consider now the proof of the addition theorem for two successive excircular sectors, of which the former starts from the principal axis. Let AOP and POQ be two such sectors (Fig. 10);

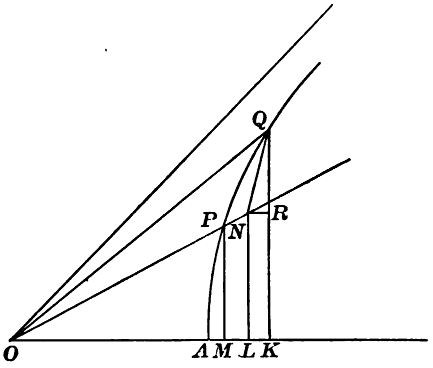

FIG. 10.

the lines PM and QK are drawn perpendicular to OA as before, but QN must now be drawn parallel to the tangent at P; NR is drawn perpendicular to QK as before. Let u denote the ratio to a^2 of twice the area of AOP, and v that of POQ. By definition,

$$\cosh u = \frac{OM}{OA} \quad \text{and} \quad \sinh u = \frac{MP}{OA}.$$

By $\cosh v$ is meant the ratio $\frac{ON}{OP}$, when the sector v is moved back so as to start from OA, the area being retained constant; and by $\sinh v$ is meant the ratio $\frac{NQ}{OP}$ under the same conditions. Now it may be shown that whatever the position of P, these ratios are constant, provided the area of the sector is constant in magnitude; hence,

$$\cosh v = \frac{ON}{OP}, \quad \sinh v = \frac{NQ}{OP}.$$

By the property of the tangent to the curve, the triangles MOP and RQN are similar as before, but now NR is positive with respect to OA. With that modification, the same proof applies as before, giving

$$\frac{OK}{OA} = \frac{OM}{OA}\frac{ON}{OP} + \frac{MP}{OA}\frac{NQ}{OP};$$

that is, $\cosh(u+v) = \cosh u \cosh v + \sinh u \sinh v$,

and $$\frac{KQ}{OA} = \frac{MP}{OA}\frac{ON}{OP} + \frac{OM}{OA}\frac{NQ}{OP};$$

that is, $\sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v$.

THE LOGARITHMIC FUNCTIONS.

The circle is a special case of the logarithmic spiral, and consequently each circular ratio is a special case of what may be called the *logarithmic* ratio. To understand this generalization it is necessary to observe (*Fundamental Theorems of Analysis generalized for Space*, p. 16), that in the case of the circle, u is not a simple scalar, but the index of an exponential expression α^u, in which α denotes the axis of the plane of the circle. In plane analysis, the α is apt to drop out of sight; but in space analysis it must be introduced explicitly, in order to distinguish one plane from another. The exponential expression α^u is equal to $e^{u\alpha^{\frac{\pi}{2}}}$, and the generalization is obtained by making the angle $\frac{\pi}{2}$ any angle w. Then

$$e^{u\alpha^w} = e^{u\cos w + u\sin w \,.\, \alpha^{\frac{\pi}{2}}}.$$

Now $u \sin w$ expresses the ratio to the square of OA of twice the area of the circular sector AOP', corresponding to the logarithmic sector AOP (Fig. 11); while $e^{u\cos w}$ denotes the manner in which the radius is lengthened.

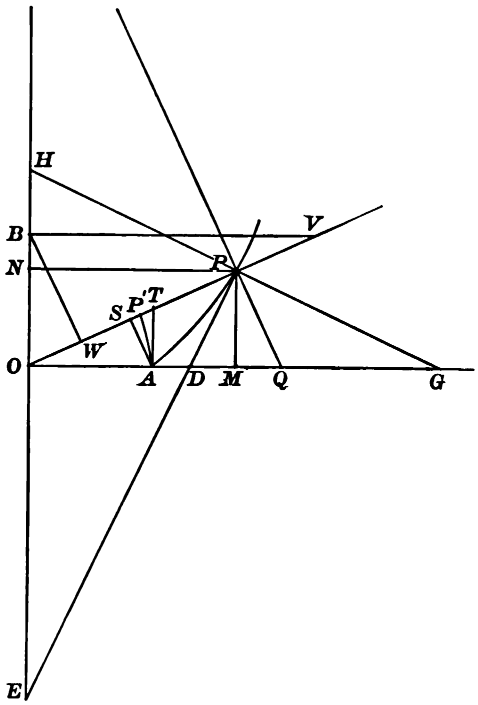

FIG. 11.

The lines PM, PQ, AT, AS, PD, PG, which refer to the axis of OA, are drawn as before; so also the complementary lines which refer to the axis of OB. The geometric definitions of the ratios are the same as before; the analytical definitions are obtained by taking the ratios of the trigonometric areas to A^2, and introducing the special condition,

$$(\cos \mathsf{AR})^2 + (\operatorname{Sin} \mathsf{AR})^2 = \mathsf{A}^4 e^{2u\cos w};$$

or,

$$\mathsf{R}^2 = \mathsf{A}^2 e^{2u\cos w}.$$

Thus,

$$\cos u,\, w = \frac{OM}{OA} \quad = \frac{\cos \mathsf{AR}}{\mathsf{A}^2},$$

$$\operatorname{Sin} u,\, w = \frac{1}{OA}\, MP = \frac{\operatorname{Sin} \mathsf{AR}}{\mathsf{A}^2},$$

$$\sin u,\, w = \frac{MP}{OA} \quad = \frac{\sin \mathsf{AR}}{\mathsf{A}^2},$$

etc., etc., etc.

The series for $\cos u, w$ is

$$1 + u\cos w + \frac{u^2}{2!}\cos 2w + \frac{u^3}{3!}\cos 3w + \text{etc.},$$

and that for $\sin u, w$ is

$$u\sin w + \frac{u^2}{2!}\sin 2w + \frac{u^3}{3!}\sin 3w + \text{etc.}$$

The values of the secant and tangent areas are deduced as before, by finding the value of DM. Now

$$DM = (\sin u,\, w)\frac{d(\cos u,\, w)}{d(\sin u,\, w)}\, OA,$$

the differentiation being with respect to u; but the ratio of the differentials does not simplify as it does in the special case of the circle.

Similarly, $GM = -\ (\sin u,\, w)\dfrac{d(\sin u,\, w)}{d(\cos u,\, w)}\, OA.$

From the areas the ratios are deduced by dividing by A^2.

When the logarithmic ratios are defined in the manner described, the addition theorem remains true. Let u, w denote the initial ratio $\frac{1}{OA}OP$ (Fig. 12), and v, w the subsequent ratio $\frac{1}{OP}OQ$. As in the case of the circle, draw QN perpendicular to

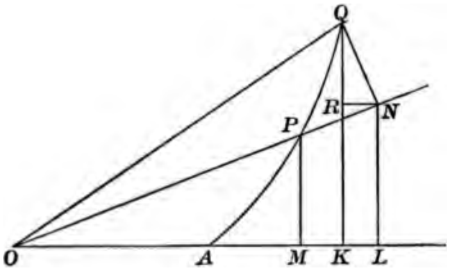

FIG. 12.

OP; PM, QK, NL perpendicular to OA; and NR perpendicular to QK. By definition,

$$\cos u, w = \frac{OM}{OA}, \quad \sin u, w = \frac{MP}{OA},$$

and
$$\cos v, w = \frac{ON}{OP}, \quad \sin v, w = \frac{NQ}{OP}.$$

Now, just as in the special case of the circle, the triangles LON and MOP are similar, and the triangles NQR and POM are similar. Hence, as before,

$$\cos \frac{1}{OA} OQ = \frac{OK}{OA} = \frac{OM}{OA}\frac{ON}{OP} - \frac{MP}{OA}\frac{NQ}{OP},$$

and
$$\sin \frac{1}{OA} OQ = \frac{KQ}{OA} = \frac{MP}{OA}\frac{ON}{OP} + \frac{OM}{OA}\frac{NQ}{OP}.$$

But the versor of $\frac{1}{OA} OQ$ is $\alpha^{u \sin w} \alpha^{v \sin w}$, that is, $\alpha^{(u+v) \sin w}$, and its ratio is $\frac{OP}{OA}\frac{OQ}{OP}$, that is, $e^{(u+v) \cos w}$. Hence $\frac{1}{OA} OQ = u + v, w$. Therefore,

$$\cos u + v, w = \cos u, w \cos v, w - \sin u, w \sin v, w$$

and
$$\sin u + v, w = \sin u, w \cos v, w + \cos u, w \sin v, w.$$

THE ELLIPTIC RATIOS.

Let the bounding line be an ellipse of which OA is the semi-major axis. The ellipse may be regarded as the orthogonal projection of a circle of radius OA upon a plane which passes through OA and makes an angle λ with the plane of the circle. Let $\cos\lambda$ be denoted by k. All lines in the circle parallel to OA remain unaltered in the projection, while all lines perpendicular to OA are diminished by the ratio $\cos\lambda$. Let A denote the area of a sector AOP of the ellipse, and as before let $u = \frac{2A}{a^2}$.

The trigonometric and the curve lines (Fig. 13) are drawn according to the same definitions as before; the geometric definitions of the ratios are the same as before. The analytical definitions of the ratios are obtained by taking the ratio of the corresponding area to A^2, and introducing the special condition that

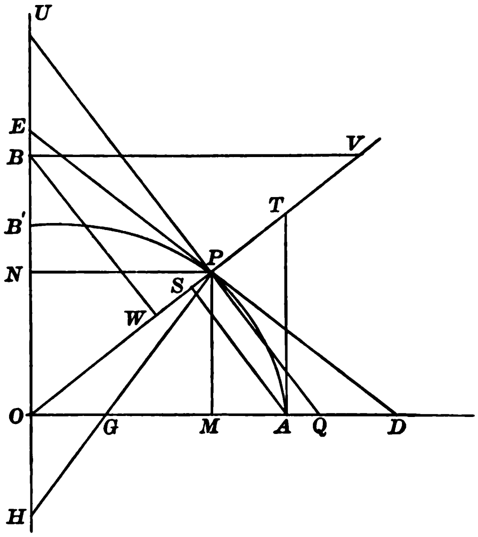

FIG. 13.

$$(\cos \mathsf{AR})^2 + \frac{(\operatorname{Sin} \mathsf{AR})^2}{k^2} = \mathsf{A}^4.$$

Thus
$$\cos u,\, k = \frac{OM}{OA} = \frac{\cos \mathsf{AR}}{\mathsf{A}^2},$$
$$\operatorname{Sin} u,\, k = \frac{1}{OA}\, MP = \frac{\operatorname{Sin} \mathsf{AR}}{\mathsf{A}^2},$$
$$\sin u,\, k = \frac{MP}{OA} = \frac{\sin \mathsf{AR}}{\mathsf{A}^2},$$
$$\text{etc.}, \qquad \text{etc.}, \qquad \text{etc.}$$

The series for the elliptic cosine is obtained by the principle that $\cos u,\, k = \cos\frac{u}{k}$, and the series for the elliptic sine by the principle that $\sin u,\, k = k \sin\frac{u}{k}$.

It is found, by application of the principle stated at p. 25, that
$$DM = -\frac{\sin^2\frac{u}{k}}{\cos\frac{u}{k}}\, OA,$$
and
$$GM = k^2 \cos\frac{u}{k}\, OA.$$

Hence
$$(OA)(OD) = \frac{\mathsf{A}^4}{\cos \mathsf{AR}},$$
$$(OA)(DP) = -\frac{1}{k^2}\,\frac{(\sin \mathsf{AR})^2}{\cos \mathsf{AR}} + \operatorname{Sin} \mathsf{AR},$$
$$(OA)(GP) = k^2 \cos \mathsf{AR} + \operatorname{Sin} \mathsf{AR},$$
$$(OA)(OG) = (1 - k^2) \cos \mathsf{AR},$$

and from these the secant, tangent, normal, and the anonymous ratio are derived by dividing by A^2.

A question arises whether the complementary ratios should be defined with respect to OB, Fig. 13, which is equal to OA, or with respect to OB', the semi-minor axis. I consider that they ought to be defined with respect to OB; the corresponding functions for OB' can be deduced from them by dividing by k.

In order to obtain the complementary curve ratios it is necessary to find *NE* and *HN*.

Now $$NE = -\cos\frac{d\sin}{d\cos}OB$$

$$= -\cos\frac{u}{k}\,\frac{d\left(k\sin\frac{u}{k}\right)}{d\left(\cos\frac{u}{k}\right)}OB$$

$$= \frac{k\cos^2\frac{u}{k}}{\sin\frac{u}{k}}OB$$

therefore $$(OB)(NE) = \frac{k^2(\cos \mathsf{AR})^2}{\sin \mathsf{AR}}$$

therefore $$(OB)(OE) = \sin \mathsf{AR} + \frac{k^2(\cos \mathsf{AR})^2}{\sin \mathsf{AR}}$$

$$= \frac{k^2\mathsf{A}^4}{\sin \mathsf{AR}},$$

and $$(OB)(EP) = -\frac{k^2(\cos \mathsf{AR})^2}{\sin \mathsf{AR}} + \operatorname{Cos} \mathsf{AR}.$$

Again, $$HN = -\cos\frac{d\cos}{d\sin}OB$$

$$= \frac{1}{k}\sin\frac{u}{k}OB$$

therefore $$(OB)(HN) = \frac{\sin \mathsf{AR}}{k^2}$$

therefore $$(OB)(HP) = \frac{\sin \mathsf{AR}}{k^2} + \operatorname{Cos} \mathsf{AR},$$

and $$(OB)(OH) = -\sin \mathsf{AR}\frac{1-k^2}{k^2}.$$

ELLIPTIC RATIOS.

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
$\alpha^{u,k}$	$\frac{1}{OA}\,OP$	$\frac{\mathrm{AR}}{\mathrm{A}^2}$
$\cos u, k$	$\frac{OM}{OA}$	$\frac{\cos \mathrm{AR}}{\mathrm{A}^2}$
$\mathrm{Sin}\, u, k$	$\frac{1}{OA}\,MP$	$\frac{\mathrm{Sin\,AR}}{\mathrm{A}^2}$
$\sin u, k$	$\frac{MP}{OA}$	$\frac{\sin \mathrm{AR}}{\mathrm{A}^2}$
$\mathrm{Sec}\, u, k$	$\frac{1}{OA}\,OT$	$\frac{\mathrm{AR}}{\cos \mathrm{AR}}$
$\sec u, k$	$\frac{OT}{OA}$	$\frac{\sqrt{\mathrm{A}^2\mathrm{R}^2}}{\cos \mathrm{AR}}$
$\mathrm{Tan}\, u, k$	$\frac{1}{OA}\,AT$	$\frac{\mathrm{Sin\,AR}}{\cos \mathrm{AR}}$
$\tan u, k$	$\frac{AT}{OA}$	$\frac{\sin \mathrm{AR}}{\cos \mathrm{AR}}$
$\mathrm{zec}\, u, k$	$\frac{OQ}{OA}$	$\frac{\mathrm{R}^2}{\cos \mathrm{AR}}$
$\mathrm{Dan}\, u, k$	$\frac{1}{OA}\,QP$	$\frac{\sin \mathrm{AR}}{\cos \mathrm{AR}}\;\frac{\mathrm{A}\sqrt{-1}\,\mathrm{R}}{\mathrm{A}^2}$
$\mathrm{dan}\, u, k$	$\frac{QP}{OA}$	$\frac{\sin \mathrm{AR}}{\cos \mathrm{AR}}\;\frac{\sqrt{\mathrm{A}^2\mathrm{R}^2}}{\mathrm{A}^2}$
$\mathrm{Gos}\, u, k$	$\frac{1}{OA}\,OS$	$\frac{\cos \mathrm{AR}}{\mathrm{R}^2}\;\frac{\mathrm{AR}}{\mathrm{A}^2}$
$\mathrm{gos}\, u, k$	$\frac{OS}{OA}$	$\frac{\cos \mathrm{AR}}{\sqrt{\mathrm{A}^2\mathrm{R}^2}}$
$\mathrm{Zin}\, u, k$	$\frac{1}{OA}\,AS$	$\frac{\sin \mathrm{AR}}{\mathrm{R}^2}\;\frac{\mathrm{A}\sqrt{-1}\,\mathrm{R}}{\mathrm{A}^2}$
$\mathrm{zin}\, u, k$	$\frac{AS}{OA}$	$\frac{\sin \mathrm{AR}}{\sqrt{\mathrm{A}^2\mathrm{R}^2}}$
$\mathrm{vers}\, u, k$	$\frac{AM}{OA}$	$-1+\frac{\cos \mathrm{AR}}{\mathrm{A}^2}$
$\mathrm{sct}\, u, k$	$\frac{OD}{OA}$	$\frac{\mathrm{A}^2}{\cos \mathrm{AR}}$

ELLIPTIC RATIOS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
Tnt u, k	$\frac{1}{OA} DP$	$-\frac{(\sin \mathsf{AR})^2}{k^2\mathsf{A}^2 \cos \mathsf{AR}} + \frac{\mathrm{Sin}\, \mathsf{AR}}{\mathsf{A}^2}$
tnt u, k	$\frac{DP}{OA}$	$\frac{\sin \mathsf{AR}}{\cos \mathsf{AR}} \frac{\sqrt{\sin^2 \mathsf{AR} + k^4 \cos^2 \mathsf{AR}}}{k^2\mathsf{A}^2}$
Nor u, k	$\frac{1}{OA} GP$	$\frac{k^2 \cos \mathsf{AR} + \mathrm{Sin}\, \mathsf{AR}}{\mathsf{A}^2}$
nor u, k	$\frac{GP}{OA}$	$\frac{\sqrt{k^4 \cos^2 \mathsf{AR} + \mathrm{Sin}^2 \mathsf{AR}}}{\mathsf{A}^2}$
co-cos u, k	$\frac{ON}{OB}$	$\frac{\sin \mathsf{AR}}{\mathsf{A}^2}$
co-Sin u, k	$\frac{1}{OB} NP$	$\frac{\mathrm{Cos}\, \mathsf{AR}}{\mathsf{A}^2}$
co-sin u, k	$\frac{NP}{OB}$	$\frac{\cos \mathsf{AR}}{\mathsf{A}^2}$
co-Sec u, k	$\frac{1}{OB} OV$	$\frac{\bar{\mathsf{A}}\mathsf{R}}{\sin \mathsf{AR}}$
co-sec u, k	$\frac{OV}{OB}$	$\frac{\sqrt{\mathsf{A}^2\mathsf{R}^2}}{\sin \mathsf{AR}}$
co-Tan u, k	$\frac{1}{OB} BV$	$\frac{\mathrm{Cos}\, \mathsf{AR}}{\sin \mathsf{AR}}$
co-tan u, k	$\frac{BV}{OB}$	$\frac{\cos \mathsf{AR}}{\sin \mathsf{AR}}$
co-zec u, k	$\frac{OU}{OB}$	$\frac{\mathsf{R}^2}{\sin \mathsf{AR}}$
co-Dan u, k	$\frac{1}{OB} UP$	$\frac{\cos \mathsf{AR}}{\sin \mathsf{AR}} \frac{\bar{\mathsf{A}}\sqrt{-1}\,\mathsf{R}}{\mathsf{A}^2}$
co-dan u, k	$\frac{UP}{OB}$	$\frac{\cos \mathsf{AR}}{\sin \mathsf{AR}} \frac{\sqrt{\mathsf{A}^2\mathsf{R}^2}}{\mathsf{A}^2}$
co-Gos u, k	$\frac{1}{OB} OW$	$\frac{\sin \mathsf{AR}}{\mathsf{R}^2} \frac{\bar{\mathsf{A}}\mathsf{R}}{\mathsf{A}^2}$
co-gos u, k	$\frac{OW}{OB}$	$\frac{\sin \mathsf{AR}}{\sqrt{\mathsf{A}^2\mathsf{R}^2}}$
co-Zin u, k	$\frac{1}{OB} BW$	$\frac{\cos \mathsf{AR}}{\mathsf{R}^2} \frac{\bar{\mathsf{A}}\sqrt{-1}\,\mathsf{R}}{\mathsf{A}^2}$

ELLIPTIC RATIOS (Continued).

FUNCTION.	GEOMETRIC DEFINITION.	ANALYTICAL DEFINITION.
co-zin u, k	$\frac{BW}{OB}$	$\frac{\cos \mathrm{AR}}{\sqrt{\mathrm{R}^2\mathrm{A}^2}}$
co-sct u, k	$\frac{OE}{OB}$	$\frac{k^2\mathrm{A}^2}{\sin \mathrm{AR}}$
co-Tnt u', k	$\frac{1}{OB}EP$	$-\frac{k^2(\cos \mathrm{AR})^2}{\mathrm{A}^2 \sin \mathrm{AR}}+\frac{\mathrm{Cos\,AR}}{\mathrm{A}^2}$
co-tnt u, k	$\frac{EP}{OB}$	$\frac{\cos \mathrm{AR}}{\mathrm{A}^2 \sin \mathrm{AR}}\sqrt{k^4\cos^2 \mathrm{AR}+\sin^2 \mathrm{AR}}$
co-Nor u, k	$\frac{1}{OB}HP$	$\frac{\sin \mathrm{AR}}{k^2\mathrm{A}^2}+\frac{\mathrm{Cos\,AR}}{\mathrm{A}^2}$
co-nor u, k	$\frac{HP}{OB}$	$\frac{\sqrt{k^4\cos^2 \mathrm{AR}+\sin^2 \mathrm{AR}}}{k^2\mathrm{A}^2}$
anon u, k	$\frac{OH}{OB}$	$-\frac{\sin \mathrm{AR}}{\mathrm{A}^2}\,\frac{1-k^2}{k^2}$

When the elliptic ratios are so defined it is not difficult to obtain the generalized addition theorem. Let AOP and POQ (Fig. 14) be two successive elliptic sectors of which the former starts from the principal axis. Draw QN parallel to the tangent at P; and PM, QK, NL perpendicular to QA, and NR perpendicular to QK. Let u denote the ratio of twice the area of the sector AOP to the square on OA, and v that of twice the area of the sector POQ to the square on OA; it follows that $u+v$ is the ratio of twice the area of the sector AOQ to the square on OA. By definition

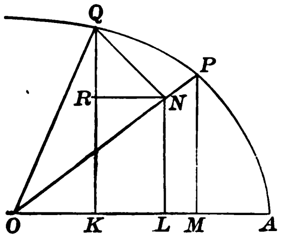

FIG. 14.

$$\cos u,\ k=\frac{OM}{OA} \qquad \sin u,\ k=\frac{MP}{OA},$$

and

$$\cos u+v,\ k=\frac{OK}{OA} \qquad \sin u+v,\ k=\frac{KQ}{OA}.$$

Now $$\cos v,\, k = \cos\frac{v}{k} = \frac{ON}{OP},$$

because the lines ON and OP have the same direction and therefore the same ratio as the corresponding lines in the circle. But as NQ and OP have different directions, and are in general lines which do not coincide with the principal axes, the relation of their ratio to $\sin\frac{v}{k}$ is more complex. It will be found by examination of the projection that

$$\frac{NQ}{OP}\sqrt{\frac{\cos^2\frac{u}{k} + k^2\sin^2\frac{u}{k}}{\sin^2\frac{u}{k} + k^2\cos^2\frac{u}{k}}} = \sin\frac{v}{k}.$$

For the sake of brevity let the radical be denoted by q. The triangle NQR is no longer similar to the triangle POM; instead of the relation

$$\frac{NR}{NQ} = -\frac{MP}{OP}$$

we have the relation

$$\frac{NR}{NQ} = -\frac{MP}{OP}\frac{q}{k}.$$

Now $$\cos u+v,\, k = \frac{OK}{OL}$$

$$= \frac{OL}{OA} + \frac{LK}{OA}$$

$$= \frac{OM}{OP}\frac{ON}{OA} - \frac{MP}{OP}\frac{NQ}{OA}\frac{q}{k}$$

$$= \frac{OM}{OA}\frac{ON}{OP} - \frac{MP}{OA}\frac{NQ}{OP}\frac{q}{k}$$

$$= \cos u,\, k \cos v,\, k - \frac{\sin u,\, k \sin v,\, k}{k^2}.$$

Again, $\sin u+v,\, k=\frac{KQ}{OA}$

$$=\frac{LN}{OA}+\frac{RQ}{OA}$$

$$=\frac{ON}{OA}\,\frac{MP}{OP}+\frac{OM}{OA}\,\frac{NQ}{OP}kq$$

$$=\frac{MP}{OA}\,\frac{ON}{OP}+\frac{OM}{OA}\,\frac{NQ}{OP}kq$$

$$=\sin u,\, k\cos v,\, k+\cos u,\, k\sin v,\, k.$$

By $\sin v,\, k$ is meant the ratio of NQ to OP when the sector is shifted back without change of area so as to start from the principal axis.

THE HYPERBOLIC RATIOS.

Let the bounding line be an hyperbola of which OA is the semi-major axis. The hyperbola may be regarded as the orthogonal projection of an excircle of radius OA upon a plane which passes through OA and makes an angle λ with the plane of the circle. As before, let $\cos\lambda$ be denoted by k. Let A denote the area of a sector of the hyperbola, and let $u=\frac{2A}{a^2}$.

The triangular and the curve lines are drawn according to the same definitions as before; the geometric definitions of the several functions of u and k are the same as before. The analytical definitions of the ratios are obtained by taking the ratio of the corresponding area to A^2, and introducing the special condition that

$$(\cos \mathsf{AR})^2-\frac{(\mathrm{Sin}\ \mathsf{AR})^2}{k^2}=\mathsf{A}^4.$$

Thus $$\cosh u,\, k=\frac{OM}{OA}=\frac{\cos \mathsf{AR}}{\mathsf{A}^2},$$

etc., etc., etc.

THE COMPLEX RATIOS.

Our method of definition applies also to the complex ratios. Let AOQ (Fig. 15) be a complex sector made up of a circular

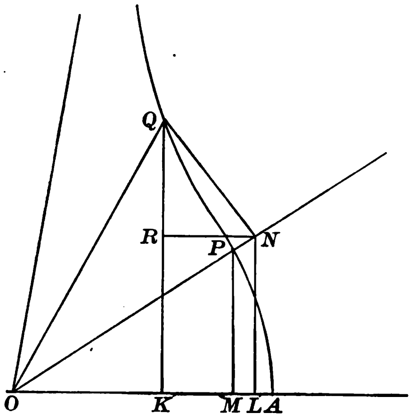

FIG. 15.

sector AOP and an excircular sector POQ. Draw QN perpendicular to OP, and PM, QK, NL perpendicular to OA, also NR perpendicular to QK. Let u denote the ratio of twice the area of AOP to the square on OA, and v that of twice the area of POQ to the square of OP. To distinguish the form of the area let i be prefixed to v; then $u+iv$ denotes the ratio of twice the area of the complex sector AOQ to the square of OA. By definition

$$\cos u=\frac{OM}{OA}, \qquad \sin u=\frac{MP}{OA},$$

$$\cos iv=\frac{ON}{OP}, \qquad \sin iv=\frac{NQ}{OP},$$

$$\cos u+iv=\frac{OK}{OA}, \qquad \sin u+iv=\frac{KQ}{OA}.$$

Now as in the case of the circle

$$\frac{OK}{OA}=\frac{OM}{OA}\cdot\frac{ON}{OP}-\frac{MP}{OA}\,\frac{NQ}{OP},$$

therefore $$\cos u+iv=\cos u\cos iv-\sin u\sin iv$$

$$=\cos u\cosh v-\sin u\sinh v.$$

Similarly $$\frac{KQ}{OA}=\frac{MP}{OA}\frac{ON}{OP}+\frac{OM}{OA}\frac{NQ}{OP}$$

$$=\sin u\cos iv+\cos u\sin iv$$

$$=\sin u\cosh v+\cos u\sinh v.$$

The function $\cos iv$ is obtained from $\cos v$ by supposing $i=\sqrt{-1}$; and $\sin iv$ from $\sin v$ by the same process, only the $\sqrt{-1}$ common to all the terms must be removed.

From the symmetry of the formulæ it is evident that the order of circular-excircular or excircular-circular is indifferent.

To avoid fine, this book should be returned on
or before the date last stamped below

MAR 9 1973

Gaylord
PAMPHLET BINDER
Syracuse, N. Y.
Stockton, Calif.

Stanford University Libraries
3 6105 002 055 452